

Newton-Hooke type symmetry of anisotropic oscillators*

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Abstract

The rotation-less Newton–Hooke - type symmetry found recently in the Hill problem and instrumental for explaining the center-of-mass decomposition is generalized to an arbitrary anisotropic oscillator in the plane. Conversely, the latter system is shown, by the orbit method, to be the most general one with such a symmetry. Full Newton-Hooke symmetry is recovered in the isotropic case.

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* Dedicated to the memory of J.-M. Souriau, deceased on March 15 2012, at the age of 90.

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1. INTRODUCTION

Renewed interest in Kohn’s theorem on decomposing a system of charged particles in a magnetic field into center-of-mass and relative coordinates stems from relating it to the Newton-Hooke symmetry of the Landau problem [1–5]. Recently, another example was found, however [6, 7] : Hill’s equations studied in Celestial Mechanics also admit a center-of-mass decomposition, but no full Newton-Hooke symmetry. Rotations are broken, but translations and (generalized) boosts are still symmetries, hinting at that it is the subgroup spanned by the latter which is important for the purpose; additional symmetries like rotations are irrelevant.

In this paper, when referring to *Newton-Hooke type* symmetry it is *Newton-Hooke with or without rotations* that we have in mind.

The possibility of decomposing an *isolated system* into center-of-mass and relative coordinates has been linked to Galilean symmetry : Souriau [8] argued, in fact, that this property depends on the Galilei group having an invariant *Abelian subgroup* — namely the one spanned by *translations* and (Galilean) *boosts*.

Remarkably, the cohomological structure which determines the existence of a central extension originates precisely in this subgroup [8]. Remember that, in dimension $d \geq 3$, both the Newton-Hooke and the Galilei groups have a one-parameter central extension, but in the plane they both admit an “exotic”, *two-parameter* central extension [9–12].

In this paper, we focus our attention at the Newton-Hooke case, the Galilean one being rather standard.

The ordinary Landau problem admits the one-parameter centrally extended Newton-Hooke group as symmetry [1], but the “exotic” [non-commutative] version has indeed the two-parameter version [4]. In the Hill case, rotation-less “Newton-Hooke type” symmetry, with one (or, in the “exotic” case, with two) central extensions could be established [6, 7].

Our main result proved in Section 5, is :

Theorem 1 : *The most general planar system with Newton-Hooke - type symmetry is a (possibly non-commutative) anisotropic oscillator in a uniform magnetic background. The symmetry extends to full Newton-Hooke symmetry in the isotropic case.*

The proof will be accomplished by applying the *orbit method*, which provides us indeed

with all systems upon which the symmetry group acts transitively [8, 13–15].

From the technical point of view, we will find it convenient to use *chiral decomposition* [6, 7, 12, 16, 17, 20]. The motion in the (ordinary) Hall effect can in fact be decomposed into two uncoupled chiral oscillators with opposite chirality [16]. Conversely, combining two 1d chiral oscillators may yield the non-commutative Landau problem [17–20]; then the chiral method allows for an elegant derivation of the (Newton-Hooke) symmetry.

Recently, the method was extended to the Hill problem [6, 7] which is in fact a “maximally anisotropic oscillator”; here we further extend it to arbitrary anisotropy.

Our paper is organized as follows.

In section 2 chiral oscillators are reviewed.

Then we outline the Newton-Hooke symmetry in the Landau- , and the rotation-less Newton-Hooke type symmetry in the Hill cases, respectively.

In Section 4, we generalize to an arbitrary, possibly anisotropic, oscillator.

In Section 5 we proceed conversely: applying the orbit method we describe all systems with Newton-Hooke-type symmetry acting transitively.

We also study the arising of further symmetries and explain the difference between symmetry with or without rotations. Our results allow us to deduce

Theorem 2 : *The system is either a truly anisotropic oscillator with Newton-Hooke-type symmetry only and no rotations, or it is isotropic with full Newton-Hooke symmetry, including rotations.*

Moreover, in the first case, it can be brought into a “Hill-type form”, and in the second one it can be transformed into a free particle, cf. Sections 5 D and 6.

An outlook is presented in the Conclusion, section 7.

2. CHIRAL OSCILLATORS

Chiral oscillators arise owing to the ambiguity of the phase-space description of a harmonic oscillator. In detail, let us consider a one-dimensional harmonic oscillator of unit mass $m = 1$ and frequency ω . Viewing the position and velocity, x and \dot{x} , simply as coordinates

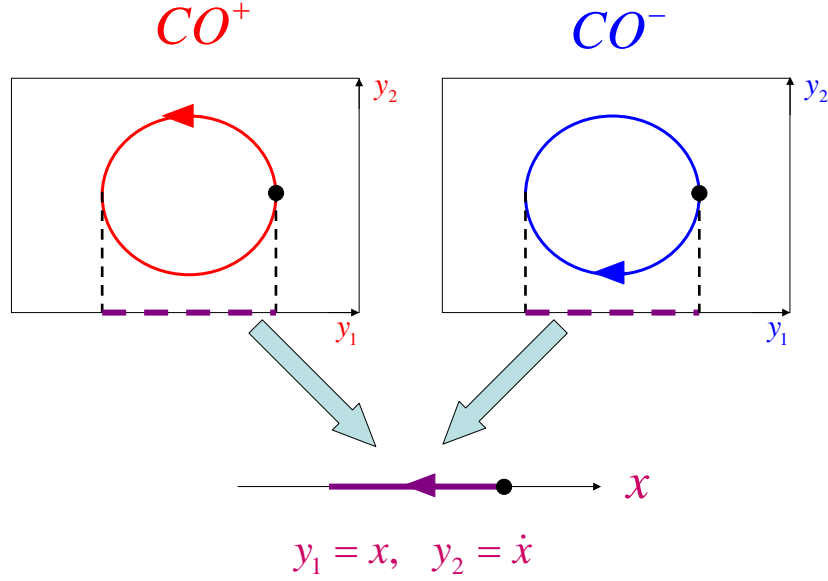


FIG. 1: The phase-space trajectory of a chiral oscillator turns clockwise or anti-clockwise, depending on the sign of the frequency. Both trajectories project, however, onto the same motion in configuration space.

on the phase space, we write

$$y_1 = x, \quad y_2 = \dot{x}, \quad (2.1)$$

and consider the two first-order phase space Lagrangians

$$L_{\pm} = \pm \frac{1}{2} \epsilon_{ij} y_i \dot{y}_j - \frac{\omega}{2} \mathbf{y}^2. \quad (2.2)$$

The associated (Euler-Lagrange) equations read

$$\dot{y}_i = \mp \omega \epsilon_{ij} y_j. \quad (2.3)$$

Our clue is that, for both signs in eqn. (2.2), eliminating either y_1 or y_2 yields, for the remaining variable, the *same* equation, namely that of a 1d harmonic oscillator,

$$\ddot{y}_i + \omega^2 y_i = 0, \quad i = 1, 2. \quad (2.4)$$

The solutions of (2.3) are simple rotations in phase space – but in *opposite* directions, depending on the sign [28]. (This is indeed the very meaning of the word “chiral”). Then we note that *both* types of motions project into configuration space according to the *same* motion $x(t)$, as illustrated on Fig. 1.

We note that the same conclusion can be reached using a Hamiltonian framework. The eqns. (2.3) are indeed those of the symplectic structure and Hamiltonian

$$\Omega_{\pm} = \pm \frac{1}{2} \epsilon_{ij} dy_i \wedge dy_j, \quad H = \frac{1}{2} \omega \mathbf{y}^2, \quad (2.5)$$

namely $\dot{y}_i = \{y_i, H\}_{\pm}$, where the Poisson brackets $\{\cdot, \cdot\}_{\pm}$ are those associated with the chosen symplectic structure Ω_{\pm} . The coordinates y_i are non-commuting,

$$\{y_1, y_2\}_{\pm} = \mp 1, \quad (2.6)$$

— as it is natural for position and momentum on the phase space. We mention for completeness that the Lagrangians (2.2) are the Cartan forms of the Souriau forms [8, 21],

$$L_{\pm} dt = \lambda_{\pm}, \quad d\lambda_{\pm} = \Omega_{\pm} - dH \wedge dt. \quad (2.7)$$

3. TWO SYSTEMS WITH NEWTON-HOOKE [TYPE] SYMMETRIES

A. Landau problem

The classical example of a system with one-parameter-centrally-extended Newton-Hooke symmetry is provided by the “ordinary” [meaning commutative] Landau problem [1]. Generalizing the latter, we consider N “exotic” particles endowed with masses, charges and non-commutative parameters m_a , e_a and θ_a , respectively, moving in a planar electromagnetic field B, \mathbf{E} [4]. Following [11, 19], we describe our system by

$$\begin{aligned} m_a^* \dot{x}_a^i &= p_a^i - m_a e_a \theta_a \varepsilon^{ij} E^j, \\ \dot{p}_a^i &= e_a B \varepsilon^{ij} \dot{x}_a^j + e_a E^i, \end{aligned} \quad (3.1)$$

where

$$m_a^* = \Delta_a m_a \quad \text{with} \quad \Delta_a = 1 - e_a \theta_a B \quad (3.2)$$

is the effective mass of the particle labeled by $a = 1, \dots, N$. Note, in the first relations, also the “anomalous velocity terms” perpendicular to the electric field, \mathbf{E} . The variables \mathbf{p}_a here *could* be called “momenta” – but to avoid confusion with the conserved quantities, we simply consider them as coordinates on the phase space.

Although our theory works for any B and \mathbf{E} , we assume, for simplicity, that the magnetic field is constant, $B = \text{const}$, and that the electric field is that of an isotropic harmonic

trap, $-k\mathbf{x}$, augmented with an interparticle force coming from some two-body potential, $V = \sum_{a \neq b} V_{ab}(\mathbf{x}_a - \mathbf{x}_b)$.

For $\theta_a = 0$, the ordinary Landau problem is recovered.

Summing over all particles, we find that when e_a/m_a and $e_a\theta_a$ are both constants i.e. when the *generalized Kohn conditions* [4],

$$\kappa_a \equiv \frac{e_a}{m_a} = \frac{e}{M} \equiv \kappa, \quad e_a\theta_a = e\Theta, \quad \Theta = \frac{\sum_a m_a^2 \theta_a}{M^2} \quad (3.3)$$

hold (where $M = \sum_a m_a$ and $e = \sum_a e_a$ are the total mass and charge), then the center-of-mass, $\mathbf{X} = \sum_a m_a \mathbf{x}_a / M$, splits off,

$$\begin{aligned} M^* \dot{X}^i &= P^i - Me\Theta \varepsilon^{ij} E^j, \\ \dot{P}^i &= eB \varepsilon^{ij} \dot{X}^j + eE^i, \end{aligned} \quad (3.4)$$

where

$$M^* = \Delta M, \quad \Delta = 1 - e\Theta B, \quad \mathbf{P} = \sum_a \mathbf{p}_a. \quad (3.5)$$

The center-of-mass behaves hence as a *single “exotic” particle* carrying the total mass, charge and non-commutative parameter, M , e and Θ , respectively.

We note that eqns. (3.4) are in fact Hamilton’s equations for the Poisson structure [19],

$$H = \frac{\mathbf{P}^2}{2M} + V(\mathbf{X}), \quad (3.6)$$

$$\{X^i, X^j\} = \frac{\Theta}{\Delta} \varepsilon^{ij}, \quad \{X^i, P^j\} = \frac{\delta^{ij}}{\Delta}, \quad \{P^i, P^j\} = \frac{eB}{\Delta} \varepsilon^{ij}. \quad (3.7)$$

When the [generalized] Kohn condition (3.3) is satisfied, then, for identical initial velocities, all individual particles move in the same way, — and this motion is shared by their center of mass, cf. Fig. 2.

The best way to understand the intuitive content of the Kohn condition is, however, to consider what happens when the Kohn condition is *not* satisfied. Consider, for example, two particles in a pure magnetic field such that

$$\kappa_2 \equiv \frac{e_2}{m_2} = 2\kappa_1 \equiv 2 \frac{e_1}{m_1}. \quad (3.8)$$

Then, assuming identical initial velocities, each of them performs a rotational motion but with *different radii*,

$$R = (m/e) \frac{v}{B} \quad \Rightarrow \quad R_2 = \frac{1}{2} R_1, \quad (3.9)$$

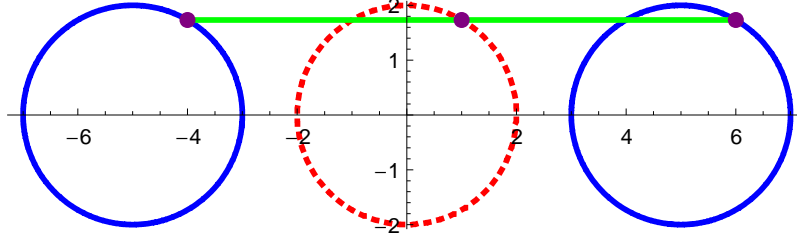


FIG. 2: If the Kohn conditions (3.3) are satisfied, all particles turn along circles of equal radii with common angular velocity. Their motion is shared by their center of mass (in dashed).

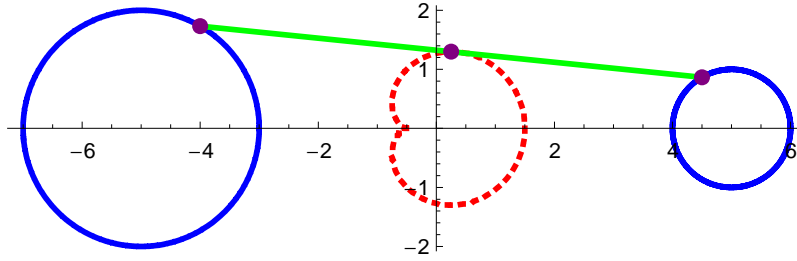


FIG. 3: If the Kohn conditions (3.3) are not satisfied, $\kappa_2 \equiv e_2/m_2 = 2e_1/m_1 \equiv 2\kappa_1$, for example, the individual radii and the angular velocities are different. The motion is not more collective, and the center of mass describes a complicated (dashed) curve.

and with *different* frequencies:

$$\omega = \frac{v}{R} \quad \Rightarrow \quad \omega_2 = 2\omega_1 \quad (3.10)$$

[so that $\omega_1 R_1 = v = \omega_2 R_2$]. Their center-of-mass would then clearly *not* move on a circle, rather on some complicated curve, cf. Fig. 3. The 3-body situation is illustrated on Fig. 4.

Symmetries.

Let us restrict ourselves henceforth to the purely magnetic case, $\mathbf{E} = 0$ and to the center-of-mass motion. The coordinate \mathbf{P} is not conserved; one readily shows, however, that the “magnetic momentum” (which can also be derived by Noether’s theorem as the conserved quantity associated with the translational symmetry [20]) [29] and “magnetic center-of-mass”,

$$\begin{aligned} \Pi_i &= M\Delta(\dot{X}_i - \omega^* \varepsilon_{ij} X_j), \\ \mathbf{K} &= M\Delta^2 R(\omega^* t) \dot{\mathbf{X}}, \end{aligned} \quad (3.11)$$

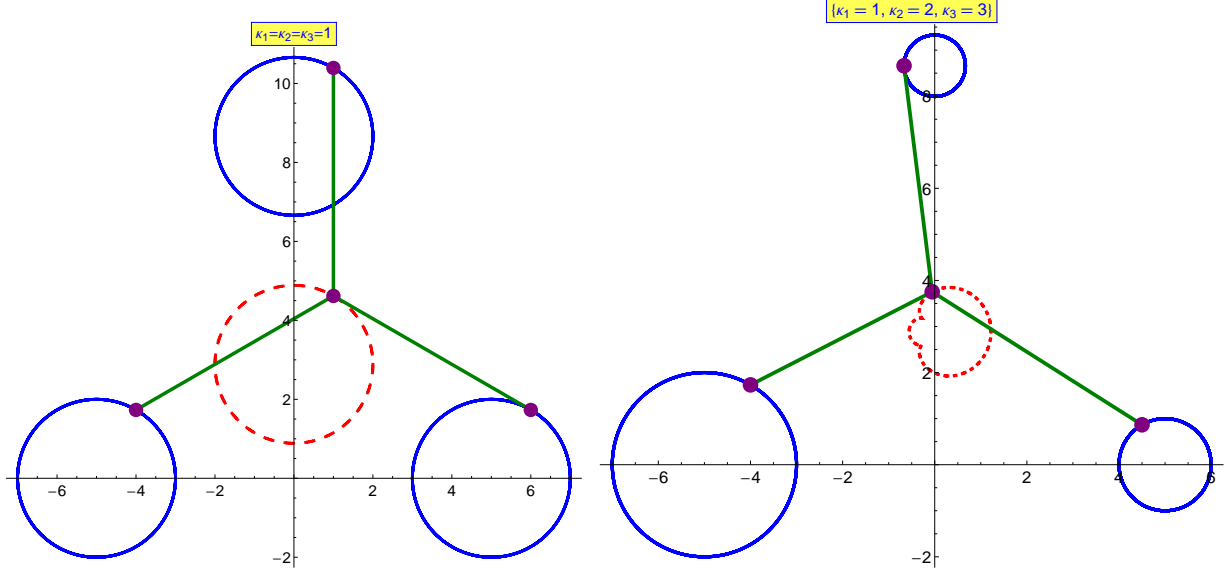


FIG. 4: The behavior of a 3-body system. (a) If the Kohn conditions (3.3) are satisfied, all particles move collectively, along with their center-of-mass. (b) If (3.3) is not satisfied, $\kappa_1 = 1, \kappa_2 = 2, \kappa_3 = 3$, for example, the motion is not more collective, and the center of mass (in dashed) describes a complicated curve.

respectively, where $\omega^* = eB/M^* = \omega/\Delta$, are both conserved [30], and span indeed two uncoupled Heisenberg algebras with central charges $-M\omega$ and $\Delta M\omega$,

$$\{\Pi^i, \Pi^j\} = -M\omega\varepsilon^{ij}, \quad \{K^i, K^j\} = \Delta M\omega\varepsilon^{ij}, \quad \{\Pi^i, K^j\} = 0. \quad (3.12)$$

Time translations and rotations are plainly symmetries also, and the associated conserved quantities, namely the Hamiltonian H in (3.6) [31], augmented with the total angular momentum [11, 19],

$$J = \mathbf{X} \times \mathbf{P} + \frac{eB}{2}\mathbf{x}^2 + \frac{\Theta}{2}\mathbf{P}^2, \quad (3.13)$$

have commutation relations

$$\{H, \Pi^i\} = 0, \quad \{H, K^i\} = -\frac{\omega}{\Delta}\varepsilon^{ij}K^j, \quad (3.14)$$

$$\{J, X^i\} = \varepsilon^{ij}X^j, \quad \{J, P^i\} = \varepsilon^{ij}P^j, \quad \{J, H\} = 0. \quad (3.15)$$

In conclusion, the exotic Landau problem [with or without an isotropic harmonic trapping force] admits an “exotic” i.e. two-parameter centrally extended Newton-Hooke symmetry [4, 20]. In the commutative case $\Theta = 0$, the central charges are correlated, $\mp M\omega$, and the symmetry reduces to the one-parameter extension studied in [1].

We record for further use that the total angular momentum, J in (3.13), can also be presented in a number of different ways. Firstly, we note that the new variables [11]

$$Q_i = x_i + \frac{1}{eB} \left(1 - \sqrt{1 - \theta eB}\right) \varepsilon_{ij} p_j, \quad (3.16)$$

$$P_i = \frac{1 + \sqrt{1 - \theta eB}}{2} p_i - \frac{1}{2eB} \varepsilon_{ij} x_j, \quad (3.17)$$

are canonical, and in their terms the total angular momentum is simply

$$J = \mathbf{Q} \times \mathbf{P}. \quad (3.18)$$

Here we just mention that using chiral coordinates (sect. 4 C), the angular momentum can further be decomposed, see (4.22).

From now on the generalized Kohn conditions (3.3) will always be assumed, allowing us to consider the center-of-mass alone. Coordinates will again be denoted by lower-case letters, as for a one-particle theory.

B. Hill problem

The center-of-mass of N celestial bodies, moving approximately along circular orbits in the gravitational field of some heavy central object, is described, in the linear approximation, by the planar equations [6],

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - 3\omega^2 x &= 0, \\ \ddot{y} + 2\omega\dot{x} &= 0, \end{aligned} \quad (3.19)$$

where ω is the angular velocity $\omega^2 = GM/R^3$ for a circular Keplerian trajectory of radius R . Note here the anisotropic oscillator term, which is the remnant of the centrifugal and Newtonian expressions under the used approximation.

The system is conveniently analyzed in terms of the chiral coordinates [6]

$$\mathbf{x} = \mathbf{X}_+ + \mathbf{X}_-, \quad p^1 = \frac{1}{2}\omega X_+^2, \quad p^2 = -2\omega X_+^1 - \frac{3}{2}\omega X_-^1. \quad (3.20)$$

Our investigations can in fact be generalized to exotic particles with $\theta \neq 0$, see [7]. Skipping details, we mention that in both [exotic or not] cases we find that ordinary translations and

certain “time dependent translations” (also called “generalized boosts”),

$$\begin{aligned}\mathbf{\Pi} &= \begin{pmatrix} X_-^1(t) \\ X_-^2(t) + \frac{3}{2}\omega t X_-^1(t) \end{pmatrix}, \\ \mathbf{K} &= \begin{pmatrix} X_+^1(t) \cos(\omega/\Delta)t - \frac{1}{2\Gamma} X_+^2(t) \sin(\omega/\Delta)t \\ 2\Gamma X_+^1(t) \sin(\omega/\Delta)t + X_+^2(t) \cos(\omega/\Delta)t \end{pmatrix},\end{aligned}\tag{3.21}$$

are conserved, where

$$\Delta = 1 - 2m\omega\theta, \quad \Gamma = 1 - 3m\theta\omega/2.\tag{3.22}$$

Their commutation relations are, once again, those of two exotic Heisenberg algebras with central charges $-(2/m\omega)$ and $(\Gamma/\Delta)(2m\omega)$, respectively,

$$\{\Pi^1, \Pi^2\} = -\frac{2}{m\omega}, \quad \{K^1, K^2\} = \frac{\Gamma}{\Delta} \frac{2}{m\omega}, \quad \{\Pi_i, K_j\} = 0.\tag{3.23}$$

In the commutative case $\Gamma = \Delta = 1$, and the one-parameter centrally extended symmetry found in [6] is recovered.

The Hamiltonian,

$$H = H_+ + H_- = \frac{m\omega^2}{2} \left(X_+^1 X_+^1 + \frac{1}{4\Gamma^2} X_+^2 X_+^2 \right) - \frac{3m\omega^2}{8} X_-^1 X_-^1,\tag{3.24}$$

is also conserved. Its commutation relations with translations and boosts read

$$\begin{aligned}\{H, \Pi^1\} &= 0, & \{H, \Pi^2\} &= \frac{3}{2}\omega \Pi^1, \\ \{H, K^1\} &= -\frac{\omega^*}{2\Gamma} K^2, & \{H, K^2\} &= 2\Gamma\omega^* K^1.\end{aligned}\tag{3.25}$$

As rotational symmetry is plainly broken, the total symmetry of the Hill problem is *Newton-Hooke* without *rotations*.

4. ANISOTROPIC HARMONIC OSCILLATOR

We note that the Hill problem is in fact a maximally anisotropic “one sided” oscillator. The case of a general anisotropic oscillator is worth studying in some detail therefore.

A. Chiral coordinates

Consistently with the general theory sketched in Section 3 A, an “exotic” [i.e., non-commutative] charged harmonic oscillator in the plane in a homogenous magnetic field B is described by the symplectic form and Hamiltonian,

$$\Omega = dp^i \wedge dx^i + \frac{\theta}{2} \varepsilon^{ij} dp^i \wedge dp^j + \frac{eB}{2} \varepsilon^{ij} dx^i \wedge dx^j, \quad (4.1)$$

$$H = \frac{\mathbf{p}^2}{2m} + V, \quad V = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2, \quad (4.2)$$

respectively, with the parameters having the same physical interpretation as before. The spring constants k_1 and k_2 may or may not be identical.

The idea of Alvarez *et al.* [17] has been to *combine* chiral oscillators. Multiplying both the symplectic form and the Hamiltonian (or, alternatively, the Lagrangian) by the same overall constant μ ,

$$\Omega \rightarrow \mu \Omega, \quad H \rightarrow \mu H \quad \text{i.e.} \quad L \rightarrow \mu L,$$

would not change the equations of motion. But what happens, if we multiply them with *different* coefficients before *adding* them ? Conversely, can we decompose a given system into two chiral parts ? To answer these questions we introduce, following [6, 17, 20], new coordinates on the phase space,

$$x^i = X_+^i + X_-^i, \quad p^1 = \alpha_+ X_+^2 + \alpha_- X_-^2, \quad p^2 = -\beta_+ X_+^1 - \beta_- X_-^1, \quad (4.3)$$

where the coefficients α_{\pm} and β_{\pm} will be determined from the requirement that both the symplectic form and the Hamiltonian should split into two uncoupled one-dimensional sub-systems we shall call chiral components. Inserting (4.3) into (4.1) shows that the symplectic form splits as $\Omega = \Omega_+ + \Omega_-$, whenever

$$\alpha_- + \beta_+ - \theta \alpha_- \beta_+ = eB, \quad \alpha_+ + \beta_- - \theta \alpha_+ \beta_- = eB. \quad (4.4)$$

Similarly, inserting (4.3) into (4.2) yields that the Hamiltonian splits as $H = H_+ + H_-$ when

$$\alpha_+ \alpha_- + m k_2 = 0, \quad \beta_+ \beta_- + m k_1 = 0. \quad (4.5)$$

Then a tedious calculation allows choosing

$$\alpha_+ = -\frac{1}{2(eB + \theta mk_1)} \left(-e^2 B^2 + m(k_2 - k_1) + \theta^2 m^2 k_1 k_2 \right. \\ \left. + \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right), \quad (4.6)$$

$$\alpha_- = \frac{1}{2(eB + \theta mk_1)} \left(e^2 B^2 - m(k_2 - k_1) - \theta^2 m^2 k_1 k_2 \right. \\ \left. + \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right), \quad (4.7)$$

and

$$\beta_+ = -\frac{1}{2(eB + \theta mk_2)} \left(-e^2 B^2 + m(k_1 - k_2) + \theta^2 m^2 k_1 k_2 \right. \\ \left. + \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right), \quad (4.8)$$

$$\beta_- = \frac{1}{2(eB + \theta mk_2)} \left(e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2 \right. \\ \left. + \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right). \quad (4.9)$$

provides us with decomposed symplectic form and the Hamiltonian,

$$\Omega = \Omega_+ + \Omega_- = \quad (4.10)$$

$$\underbrace{(-\alpha_+ - \beta_+ + \theta\alpha_+\beta_+ + eB)}_{\mu_+} dX_+^1 \wedge dX_+^2 + \underbrace{(-\alpha_- - \beta_- + \theta\alpha_-\beta_- + eB)}_{\mu_-} dX_-^1 \wedge dX_-^2,$$

and

$$H = H_+ + H_- = \frac{1}{2m} \times \quad (4.11)$$

$$\left[(\beta_+^2 + mk_1) X_+^1 X_+^1 + (\alpha_+^2 + mk_2) X_+^2 X_+^2 + (\beta_-^2 + mk_1) X_-^1 X_-^1 + (\alpha_-^2 + mk_2) X_-^2 X_-^2 \right],$$

respectively.

- For $\theta = 0$ the commutative cases [6, 17] are recovered;
- For $k_1 = -3m\omega^2$, $k_2 = 0$ and $B = 2\omega$, we get

$$\begin{aligned}\alpha_+ &= 0, & \alpha_- &= \frac{m\omega}{2\Gamma}, & \beta_+ &= \frac{3}{2}m\omega, & \beta_- &= 2m\omega, \\ \mu_+ &= \frac{m\omega}{2}, & \mu_- &= -\frac{\Delta}{\Gamma} \frac{m\omega}{2},\end{aligned}\tag{4.12}$$

and the results found before in the Hill Problem [6, 7] are obtained;

- When $k_1 = k_2$ our oscillator is *isotropic*. Then $\alpha_{\pm} = \beta_{\pm}$, and (4.10-4.11) reduce to the chiral decomposition for the [exotic] Landau problem with harmonic force, studied in [20];
- For $k_1 = k_2 = 0$ the oscillator is switched off, and the system reduces to the purely-magnetic non-commutative Landau problem [17, 19, 20].

B. Motions

Let us assume that none of the coefficients μ_{\pm} vanishes. Then it follows from (4.11) that our chiral coordinates satisfy the Poisson bracket relations

$$\{X_+^i, X_+^j\} = -\frac{1}{\mu_+} \varepsilon^{ij}, \quad \{X_+^i, X_-^j\} = 0 \quad \{X_-^i, X_-^j\} = -\frac{1}{\mu_-} \varepsilon^{ij}.\tag{4.13}$$

The equations of motion read therefore

$$\begin{aligned}m\mu_{\pm}\dot{X}_{\pm}^1 &= -(\alpha_{\pm}^2 + mk_2)X_{\pm}^2, \\ m\mu_{\pm}\dot{X}_{\pm}^2 &= (\beta_{\pm}^2 + mk_1)X_{\pm}^1.\end{aligned}\tag{4.14}$$

Both chiral components X_{\pm} are governed, hence, by uncoupled equations which are reminiscent of those of 1d harmonic oscillators, to which they reduce, however, only in the isotropic case, $k_1 = k_2$.

Assuming $\alpha_{\pm}^2 + mk_2 \neq 0$ [32], eqns. (4.14) are solved by

$$\begin{aligned}X_{\pm}^1 &= A_{\pm} \cos \omega_{\pm} t + B_{\pm} \sin \omega_{\pm} t, \\ X_{\pm}^2 &= F_{\pm} \left(A_{\pm} \sin \omega_{\pm} t - B_{\pm} \cos \omega_{\pm} t \right), \quad F_{\pm} = \sqrt{\frac{\beta_{\pm}^2 + mk_1}{\alpha_{\pm}^2 + mk_2}},\end{aligned}\tag{4.15}$$

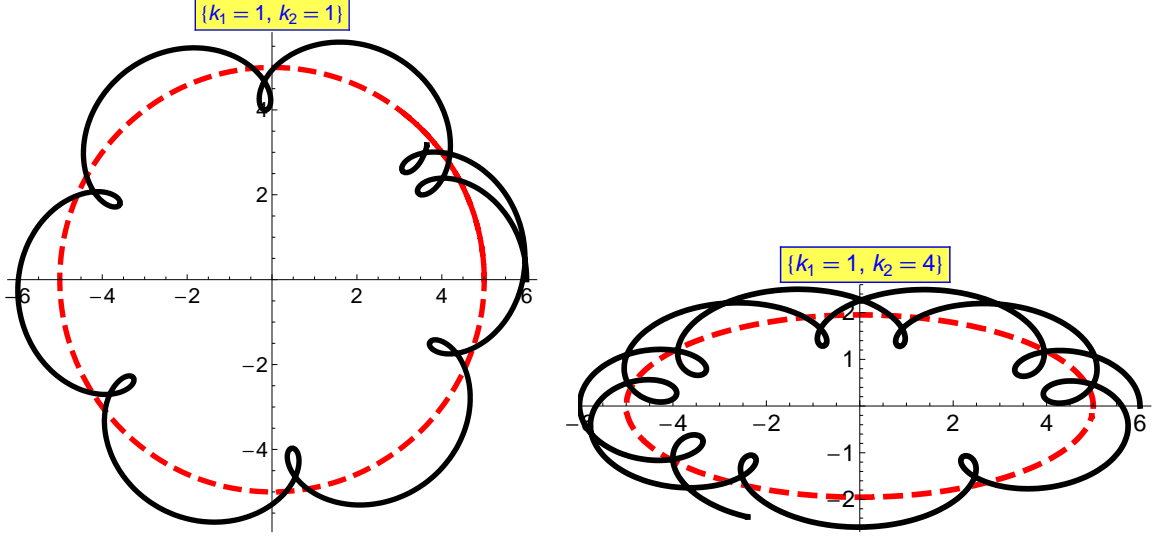


FIG. 5: “Epicyclic” motion of an oscillator in a constant magnetic field. (a) is the isotropic case $k_1 = k_2$, and (b) is an anisotropic case with $k_1 = 4k_2$. The black line is the physical trajectory, $\mathbf{x} = \mathbf{X}_+ + \mathbf{X}_-$, and the dotted red line is the motion of the guiding center, $\mathbf{X}_+(t)$.

where the frequencies read

$$\omega_{\pm} = \frac{\sqrt{(\alpha_{\pm}^2 + mk_2)(\beta_{\pm}^2 + mk_1)}}{m\mu_{\pm}}. \quad (4.16)$$

Both \mathbf{X}_{\pm} -trajectories are ellipses, as illustrated in Fig. 5. Note that the frequencies, ω_+ and ω_- are in general different even in the isotropic case, and the curves do not close therefore.

C. Symmetries

Eqns. (4.15) allow us to infer that

$$\begin{aligned} A_{\pm} &= X_{\pm}^1 \cos \omega_{\pm} t + \frac{1}{F_{\pm}} X_{\pm}^2 \sin \omega_{\pm} t, \\ B_{\pm} &= X_{\pm}^1 \sin \omega_{\pm} t - \frac{1}{F_{\pm}} X_{\pm}^2 \cos \omega_{\pm} t \end{aligned} \quad (4.17)$$

are conserved. A direct calculation yields, furthermore, for both labels \pm , the uncoupled Heisenberg algebra relations

$$\{A_{\pm}, B_{\pm}\} = -\frac{1}{F_{\pm}\mu_{\pm}}, \quad \left\{(\cdot)_{+}, (\cdot)_{-}\right\} = 0. \quad (4.18)$$

Adding the Hamiltonian (4.11), the doubly-centrally-extended rotation-less Newton-Hooke algebra is obtained.

Both sets of chiral coordinates \mathbf{X}_\pm describe $2d$ symplectic vectorspaces. The symplectic forms Ω_\pm are plainly symmetric under phase-space chiral rotations, $\mathbf{X}_\pm \rightarrow R(\mathbf{X}_\pm)$. None of the Hamiltonians H_\pm is symmetric in general, though. The natural diagonal action,

$$\mathbf{x} = \mathbf{X}_+ + \mathbf{X}_- \rightarrow R(\mathbf{X}_+) + R(\mathbf{X}_-) = R(\mathbf{x}), \quad (4.19)$$

is *not* a symmetry therefore : *rotations are broken by the anisotropy*.

In the *isotropic case*,

$$k_1 = k_2, \quad (4.20)$$

however, we have $\alpha_\pm = \beta_\pm$ and the coefficients of the quadratic terms both in H_+ and H_- are hence identical, so that the chiral rotations $\mathbf{X}_\pm \rightarrow R(\mathbf{X}_\pm)$ do act as symmetries for the components : *rotational symmetry is restored*. The square-root factors in (4.15) become unity, $F_\pm = 1$, and the trajectories become circles. The frequencies,

$$\omega_\pm = \frac{\alpha_\pm^2 + mk}{m\mu_\pm}, \quad (4.21)$$

are not identical, though, since $\alpha_+ \neq \alpha_-$ and $\mu_+ \neq \mu_-$ in general, cf. (4.6) – (4.7) and (4.10).

It is worth recording that, in terms of chiral coordinates, the total angular momentum, (3.13), is also decomposed, as

$$J = J_+ + J_-, \quad J_+ = \frac{eB}{2} \left(\vec{X}_+ \right)^2, \quad J_- = -\Delta \frac{eB}{2} \left(\vec{X}_- \right)^2, \quad (4.22)$$

where $\Delta = 1 - eB\theta$, as before. Its conservation, $\dot{J} = 0$, can also be checked directly, using the equations of motion.

We just mention that the singular case $\mu_+ = 0$ or $\mu_- = 0$, leading to Hall-type motion, can be dealt with as in the previous occasions, [4, 7, 11, 19, 20].

5. SYSTEMS WITH PRESCRIBED NH-TYPE SYMMETRY

We wish to show now that the results of the previous Sections fit, in fact, into a general framework. To this end, we assume that the dynamics under consideration is invariant under the transitive action of some Lie group G , and then classify all such symplectic manifolds upon which G acts by symplectically. The proper tool for doing this is provided by the *orbit method* [8, 13, 14].

Our choice for the group G is dictated by the following considerations. As far as possible, we would like to allow for a generalization which includes both the Galilei and the Newton-Hooke groups, and also the “rotation-free part” of the latter, considered in Sect. 3. The main characteristic features are therefore the following:

(i) there exists generators (namely of boosts and momenta) which, via the orbit method, yield the basic canonical variables;

(ii) The Hamiltonian equations of motion are linear in the latter variables; the Hamiltonian belongs therefore to the Lie algebra itself, and acts linearly on the remaining variables.

We want our generalization to be a minimal one in that no further symmetry generators beyond the above ones should be included. Such generators will appear later however for specific values of the structure constants.

Guided by these considerations, we start with the following Lie algebra commutation relations,

$$\begin{aligned}
[\xi_i, \xi_j] &= i\omega_{ij}M, \quad i, j = 1, \dots, 2N \\
[M, \xi_i] &= 0, \\
[M, H] &= 0, \\
[H, \xi_i] &= iA_{ij}\xi_j,
\end{aligned} \tag{5.1}$$

where $\omega = (\omega_{ij})$ is a non-singular antisymmetric matrix. The only non-trivial Jacobi identity,

$$[[H, \xi_i], \xi_j] + (\text{cyclic}) = 0, \tag{5.2}$$

yields the constraint $A_{ik}\omega_{kj} - A_{jk}\omega_{ki} = 0$, i.e., that $B = A\omega$ is a symmetric matrix, $B^T = B$.

The algebra (5.1) admits the Casimir operator of the form

$$C = MH - \frac{1}{2}X_{ij}\xi_i\xi_j, \tag{5.3}$$

where without loss of generality we can assume that $X = (X_{ij})$ is symmetric. C commutes with all generators, provided $A = -\omega X$.

Collecting our results, our algebra reads

$$\begin{aligned}
[\xi_i, \xi_j] &= i\omega_{ij}M, \\
[M, (\cdot)] &= 0, \\
[H, \xi_i] &= -i\omega_{ik}X_{kj}\xi_j, \\
C &= MH - \frac{1}{2}X_{ij}\xi_i\xi_j,
\end{aligned} \tag{5.4}$$

and is uniquely defined by choosing the non-singular antisymmetric matrix ω and the symmetric matrix X .

The next step is to classify the inequivalent algebras (5.4). Under the invertible transformation

$$\xi_i' = D_{ij}\xi_j, \quad \det (D_{ij}) \neq 0, \quad (5.5)$$

The matrices ω and X transform according to

$$\omega' = D\omega D^T, \quad X' = (D^{-1})^T X D^{-1}. \quad (5.6)$$

Using the latter we can find the “canonical” form in any class of equivalent algebras (5.4).

In what follows we shall restrict ourselves to the case $2N = 4$, the generalization to arbitrary N being straightforward.

To complete our classification scheme some further assumptions on the matrix X have to be made. The existence of the Casimir operator C implies that, on each orbit, the Hamiltonian is a quadratic function of the basic canonical variables, to which a trivial term, representing the internal energy, has been added (see Appendix B). Whether the energy is positive definite or not depends, therefore, on the choice of X .

The following cases will be considered separately.

A. X Positive definite

Consider first the case of a positive definite matrix X . By an appropriate choice of D in eqns. (5.6), $X = I$ can be achieved. In fact, X , being symmetric, can be diagonalized by a suitable orthogonal transformation. Then an additional diagonal transformation reduces X to the unit matrix.

Assuming $X = I$, we still have some residual transformations left at our disposition. Namely, as it is seen from eqns. (5.6), D can be taken to be an arbitrary orthogonal matrix, without spoiling the condition $X = I$. The question is now to classify all antisymmetric 4×4 matrices ω up to an orthogonal transformation. This problem is solved in Appendix A (which is actually the Euclidean version of the classification problem for electromagnetic field configurations under the action of the Lorentz group). As shown in Appendix A, ω can

be put into the form

$$\omega = \begin{pmatrix} & & \Omega_1 & 0 \\ & 0 & & \\ & & 0 & \Omega_2 \\ -\Omega_1 & 0 & & \\ 0 & -\Omega_2 & & 0 \end{pmatrix}, \quad \Omega_{1,2} > 0. \quad (5.7)$$

Defining

$$B_1 = \Omega_1^{-1}\xi_1, \quad B_2 = \Omega_2^{-1}\xi_2, \quad P_1 = \xi_3, \quad P_2 = \xi_4, \quad (5.8)$$

one finds the following non-trivial commutators :

$$\begin{aligned} [B_i, P_k] &= i\delta_{ik}M, \\ [H, B_i] &= -iP_i, \\ [H, P_i] &= i\Omega_i^2 B_i, \end{aligned} \quad (5.9)$$

together with

$$C = MH - \frac{1}{2}(P_1^2 + P_2^2 + \Omega_1^2 B_1^2 + \Omega_2^2 B_2^2). \quad (5.10)$$

Orbits

We can now apply the orbit method (Appendix B). Consider the coadjoint orbit parametrized by $m > 0$, the coordinate in dual space corresponding to the Casimir operator M and interpreted as the *mass*, and by ϵm , corresponding to the Casimir operator C and interpreted as the *internal energy*. Let $p_i, b_i, h, i = 1, 2$ be the relevant coordinates in the space dual to the Lie algebra (5.9). As shown in Appendix B, the points of the coadjoint orbit are parametrized by p_i and b_i . Defining

$$q_i = b_i/m, \quad (5.11)$$

we find

$$\{q_i, p_k\} = \delta_{ik}, \quad h = \left(\frac{p_1^2}{2m} + \frac{m\Omega_1^2}{2} q_1^2 \right) + \left(\frac{p_2^2}{2m} + \frac{m\Omega_2^2}{2} q_2^2 \right) + \epsilon. \quad (5.12)$$

Hence, we arrive at an in general *anisotropic oscillator*, as the most general system with positive definite energy, admitting the symmetry defined by the rotation-less Newton-Hooke commutation relations (5.1).

B. X semi-positive

Let us consider the case when the matrix X is semidefinite. We restrict ourselves to X having a single zero eigenvalue (as in the Hill case). Then one can select the matrix D in (5.6) in such a way that X acquires the form

$$X = \begin{pmatrix} 1 & 0 & & \\ & 0 & 0 & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}. \quad (5.13)$$

One can show again (see Appendix A) that the residual freedom in the choice of the basis of our algebra allows us to put ω into the form (5.7). Using again eqns (5.8), we find therefore

$$\begin{aligned} [H, B_i] &= -iP_i, \\ [H, P_1] &= i\Omega_1^2 B_1, \quad [H, P_2] = 0, \\ C &= MH - \tfrac{1}{2}(P_1^2 + P_2^2 + \omega_1^2 B_1^2) \end{aligned} \quad (5.14)$$

The orbit method yields, in this case, the dynamics describing a *harmonic oscillator in one direction, and free motion in the second one* — as in the Hill problem [6].

The case of multiple null eigenvalues of X can be dealt with similarly.

C. X indefinite

Let us drop, finally, the assumption of positive (semi)definiteness of X . We consider in more detail the cases of two positive – one negative – one null eigenvalues. By an appropriate choice of D one can achieve

$$X = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}, \quad \text{where} \quad G = \text{diag}(-1, 1, 1). \quad (5.15)$$

According to the results in Appendix A, the symplectic form ω can acquire three canonical forms, namely those presented in eqns (A.5) - (A.16) - (A.17). Then the orbit method gives the following dynamical systems :

(i)

$$\{q_i, p_k\} = \delta_{ik}, \quad (5.16)$$

$$h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \frac{m\Omega_2^2}{2} q_2^2 \right) + \epsilon; \quad (5.17)$$

(ii)

$$\{q_i, q_j\} = \sigma \epsilon_{ij}, \quad \{p_i, p_j\} = \tau \epsilon_{ij}, \quad (5.18)$$

$$h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \frac{q_2^2}{2} \right) + \epsilon; \quad (5.19)$$

(iii)

$$\{q_i, q_j\} = \sigma \epsilon_{ij}, \quad \{q_2, p_2\} = 1, \quad \{p_i, p_j\} = \tau \epsilon_{ij}, \quad (5.20)$$

$$h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \tau^2 \frac{q_2^2}{2m} \right) + \epsilon. \quad (5.21)$$

The parameter σ here is a clear indication of *non-commuting nature* of the coordinates q_1 and q_2 [33].

The case of non(semi)definite Hamiltonian is the most involved one. Unlike in the previous cases, after the “canonical” Hamiltonian is fixed, there still remain three inequivalent forms of the basic Poisson brackets.

The reason for that is clearly seen from the derivation given in Appendix A. The 3×3 submatrix ω_g of the matrix ω transforms, under the transformations leaving the form of the Hamiltonian invariant, as an $O(2, 1)$ antisymmetric tensor. Its canonical form depends therefore on the value of the “electromagnetic” invariant

$$\sum_{i=1}^2 (\omega_{0i})^2 - (\omega_{12})^2. \quad (5.22)$$

Depending on its value, the basic Poisson brackets can take different, inequivalent forms (assuming the form of Hamiltonian is fixed). The labeling of variables in equations (5.16) - (5.21) is dictated by our preference for the form of the Hamiltonian, rather than that of the Poisson brackets. It must be stressed, however, that the final choice of appropriate variables should be dictated by additional assumptions, not resulting from symmetry considerations only.

As an example, let us consider the planar Hill equations, as presented in Refs. [6, 7]. The Hamiltonian reads

$$H = \frac{1}{2m}(p_1^2 + p_2^2) - \frac{3m\omega^2}{2}q_2^2, \quad (5.23)$$

and yields Hill's equations for the following Poisson brackets,

$$\{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 2m\omega\epsilon_{ij}, \quad (5.24)$$

where the commutative case, $\sigma = 0$, has been chosen for simplicity. The other parameter is $\tau^2 = 3m^2\omega^2$, and $B = 2m\omega$ is the effective “magnetic” field. Let us put

$$\xi_1 = \lambda q_1, \quad \xi_2 = \sqrt{3m\omega} q_2, \quad \xi_3 = \frac{p_1}{\sqrt{m}}, \quad \xi_4 = \frac{p_2}{\sqrt{m}} \quad (5.25)$$

with $\lambda \neq 0$ arbitrary. Then H acquires the standard form

$$H = \frac{1}{2}(\xi_3^2 + \xi_4^2 - \xi_2^2), \quad (5.26)$$

and the relevant Poisson brackets read

$$\{\xi_2, \xi_4\} = \sqrt{3}\omega, \quad \{\xi_2, \xi_3\} = 2\omega. \quad (5.27)$$

Therefore one finds, with the notations of Appendix A,

$$\omega_{01} = 0, \quad \omega_{02} = \sqrt{3}\omega, \quad \omega_{12} = 2\omega, \quad \vec{\omega}^2 - \omega_{12}^2 = -\omega^2 < 0. \quad (5.28)$$

According to the classification given in Appendix A, we are dealing with the case (A.14), and the “canonical” form of the Poisson brackets is given by eqn. (A.16), in full agreement with the results of Refs. [6],[7].

D. Additional symmetries

We now study the question of additional symmetries. Consider the case of a positive definite Hamiltonian. As it has been shown in the previous Section, the initial algebra can be put into the form

$$[\xi_i, \xi_j] = i\omega_{ij}M, \quad (5.29)$$

$$[H, \xi_i] = -i\omega_{ij}\xi_j, \quad (5.30)$$

$$[M, \cdot] = 0, \quad (5.31)$$

where ω is given by eqn. (5.7). We add a new generator J which is assumed to obey

$$[J, M] = 0, \quad [J, H] = 0, \quad [J, \xi_i] = i j_{ik} \xi_k, \quad (5.32)$$

where $j = (j_{ik})$ is an appropriate matrix. The two additional Jacobi identities

$$[J, [H, \xi_i]] + (\text{cycl}) = 0 \quad [\xi_i, [J, \xi_j]] + (\text{cycl}) = 0 \quad (5.33)$$

yield $j\omega + \omega j^T = 0$, $j\omega - \omega j = 0$. Hence $j = -j^T$, and the general solution reads

(i) $\Omega_1 \neq \Omega_2$,

$$j = \alpha \begin{pmatrix} & 1 & 0 \\ 0 & & 0 \\ -1 & 0 & \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} & 0 & 0 \\ 0 & & 1 \\ 0 & 0 & \\ 0 & -1 & 0 \end{pmatrix}, \quad (5.34)$$

(ii) $\Omega_1 = \Omega_2$,

$$j = \alpha \begin{pmatrix} & 1 & 0 \\ 0 & & 0 \\ -1 & 0 & \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} & 0 & 0 \\ 0 & & 1 \\ 0 & 0 & \\ 0 & -1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} & 0 & 1 \\ 0 & & 1 \\ 0 & -1 & \\ -1 & 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (5.35)$$

Before explaining the meaning of the particular solutions, let us note that, once the equations (5.33) are obeyed, there exists a second Casimir operator, namely

$$\tilde{C} = MJ - \frac{1}{2} Y_{ij} \xi_i \xi_j, \quad (5.36)$$

where $Y = -\omega^{-1}j$. If the coadjoint orbit is parametrized by the value $m\tilde{\sigma}$ of the Casimir \tilde{C} , eqn. (5.36) [34] yields the expression for J which, as in the case of the Hamiltonian, consists of the sum of the quadratic term plus the “internal” contribution to J ,

$$mJ = \frac{1}{2} Y_{ij} z_i z_j + m\tilde{\sigma}, \quad (5.37)$$

where the z ’s are the basic variables parametrizing the points on the orbit (cf. (B.1) in Appendix B). Eqn. (5.37) allows us interpret $\tilde{\sigma}$ as the *internal angular momentum*, analogous to the *internal energy*, ϵ , in our previous formulæ, cf. [8]. Using the general solution for j , eqns. (5.34) and (5.35), one finds the following generators as functions on the coadjoint orbit (up to an internal part) :

(i) for $\Omega_1 \neq \Omega_2$,

$$H_1 = \frac{p_1^2}{2m} + \frac{m\Omega_1^2}{2}q_1^2, \quad H_2 = \frac{p_2^2}{2m} + \frac{m\Omega_2^2}{2}q_2^2; \quad (5.38)$$

(ii) for $\Omega_1 = \Omega_2 = \Omega$,

$$H_1 = \frac{p_1^2}{2m} + \frac{m\Omega^2}{2}q_1^2, \quad H_2 = \frac{p_2^2}{2m} + \frac{m\Omega^2}{2}q_2^2, \quad (5.39)$$

$$J = q_1p_2 - q_2p_1, \quad Z = p_1p_2 + m^2\Omega^2q_1q_2.$$

The meaning of the above expressions is clear. First of all, for an anisotropic oscillator we have two integrals, corresponding to the partial energies; our system is integrable.

For equal frequencies, the dynamics is superintegrable : there are three functionally independent integrals. One can choose the angular momentum as the third one. The four integrals in eqn. (5.39) are linearly independent but they are functionally dependent. Note also that our integrals (5.39), being quadratic in canonical variables, form the $u(2)$ Lie algebra — the well-known dynamical algebra of a two-dimensional isotropic harmonic oscillator. In fact, if one defines

$$V_1 \equiv \frac{1}{2}J, \quad V_2 \equiv \frac{1}{2\Omega}(H_2 - H_1), \quad V_3 \equiv \frac{1}{2m\Omega}(p_1p_2 + m^2\Omega^2q_1q_2), \quad (5.40)$$

the resulting Poisson brackets algebra reads

$$\{V_i, V_j\} = \epsilon_{ijk}V_k, \quad (5.41)$$

i.e., span the $su(2)$ algebra. The fourth generator, namely the Hamiltonian,

$$V_0 \equiv H, \quad (5.42)$$

can also be added [17]. V_0 commutes with all other V 's, completing the $su(2)$ algebra into the unitary algebra $u(2)$.

Let us remark that even for $\Omega_1 \neq \Omega_2$ there exists an additional integral, provided the ratio of the frequencies is rational,

$$r = \Omega_1/\Omega_2 = m/n. \quad (5.43)$$

It is, however, no longer quadratic in the canonical variables, yielding a W -algebra, instead of a Lie algebra [22]. In fact, an additional integral of the motion which yields our system

superintegrable can be constructed as follows. One defines the classical counterparts of the creation/annihilation operators by

$$a_i = q_i - \frac{ip_i}{m\Omega_i}, \quad \bar{a}_i = q_i + \frac{ip_i}{m\Omega_i}. \quad (5.44)$$

It is then easy to check that

$$C^{n,m} = (a_1)^n (a_2)^m \quad (5.45)$$

is an *integral of the motion*. In the isotropic case $n = m = 1$, for example,

$$C \equiv C^{1,1} = \frac{Z}{m^2\omega^2} + \frac{i}{m\Omega} J \quad (5.46)$$

is a combination of those conserved quantities in the second line of (5.39), namely of the angular momentum and the “mixed” quantity denoted by Z .

The integral $C^{n,m}$ is functionally independent of the partial energies, $H_{1,2}$. Moreover, there are no further independent (and explicitly time-independent) integrals; therefore, the Poisson bracket between $H_{1,2}$ and $C^{n,m}$ are functionally expressible in terms of them, and form a finite W -algebra [23].

Let us conclude this section with some remarks. We have shown, at least in the case of (semi)definite hamiltonian, that there exists a unique “canonical” form of the underlying dynamics. However, the choice of this canonical form is dictated by mathematical simplicity rather than by physical requirements which are, in fact, additional assumptions. It seems reasonable to assume, generally, that the physical variables are those which convert the system into (non-commutative) anisotropic oscillator in a uniform magnetic background. This can be always done because our canonical form may be converted back into any other hamiltonian form obeying the symmetry assumptions. Therefore, we end up with Theorem 1, as stated in the Introduction.

6. THE BARGMANN POINT OF VIEW

The NH symmetry of an isotropic oscillator can conveniently be derived by “importing” the Galilei symmetry of a free particle using Niederer’s transformation, which maps every half period of the oscillator onto a free particle [3, 24]. One way of seeing this is to work within Duval’s “Bargmann” framework, where classical non-relativistic motions are null

geodesics of a suitable relativistic spacetime [25, 26]. Null geodesics are invariant w.r.t. conformal transformations, and Niederer’s transformation,

$$T = \frac{\tan \omega t}{\omega}, \quad \vec{X} = \frac{\vec{x}}{\cos \omega t}, \quad S = s - \frac{\omega r^2}{2} \tan \omega t \quad (6.1)$$

maps indeed every half oscillator period conformally onto the space-time which describes a free particle,

$$d\mathbf{X}^2 + 2dTdS = \frac{1}{\cos^2 \omega t} (d\mathbf{x}^2 + 2dtds - \omega^2 r^2 dt^2). \quad (6.2)$$

This trick can *not* work for an anisotropic oscillator, though, otherwise the latter would also carry a full NH symmetry including rotation.

An anisotropic oscillator is described by the metric [35]

$$d\mathbf{x}^2 + 2dtds - (\omega_1^2 x_1^2 + \omega_2^2 x_2^2) dt^2. \quad (6.3)$$

Applying Niederer’s transformation (6.1) i.e.

$$t = \frac{\arctan \omega T}{\omega}, \quad \mathbf{x} = \frac{\mathbf{X}}{\sqrt{1 + \omega^2 T^2}}, \quad s = S + \frac{1}{2} \frac{\omega^2 \mathbf{X}^2 T}{1 + \omega^2 T^2} \quad (6.4)$$

with some ω then yields

$$\frac{1}{1 + \omega^2 T^2} \left(d\mathbf{X}^2 + 2dTdS - \frac{\omega_1^2 - \omega^2}{(1 + \omega^2 T^2)^2} X_1^2 dT^2 - \frac{\omega_2^2 - \omega^2}{(1 + \omega^2 T^2)^2} X_2^2 dT^2 \right).$$

Now choosing either $\omega = \omega_1$ or $\omega = \omega_2$ eliminates *one*, but not *both* oscillator terms, leaving us with

$$\begin{aligned} d\bar{s}^2 &= \frac{1}{1 + \omega_2^2 T^2} \left(d\mathbf{X}^2 + 2dTdS - \frac{\omega_1^2 - \omega_2^2}{(1 + \omega_2^2 T^2)^2} X_1^2 dT^2 \right) \\ &= \frac{1}{1 + \omega_1^2 T^2} \left(d\mathbf{X}^2 + 2dTdS - \frac{\omega_2^2 - \omega_1^2}{(1 + \omega_1^2 T^2)^2} X_2^2 dT^2 \right). \end{aligned} \quad (6.5)$$

[where we should have put indices 1 or 2 on \mathbf{X} , depending on our choice of ω]. For both choices we get, hence, a maximally anisotropic “one-sided” “Hill-type” system, with Newton-Hooke symmetry — *except* in the isotropic case

$$\omega_1 = \omega_2, \quad (6.6)$$

when *both* oscillator terms drop out, leaving us with a free system carrying its *full Galilei symmetry*. The latter can then be “re-imported” through the inverse of the Niederer transformation (6.1) to yield full Newton-Hooke symmetry.

In conclusion, the “prototype system” is of the “Hill type”, with its rotation-less Newton-Hooke symmetry — which, in the isotropic case, degenerates to a free particle with restored rotational symmetry.

7. CONCLUSION

Souriau [8] attributes the center-of-mass decomposition of a *free* non-relativistic system to *Galilei symmetry*, more precisely, to an invariant Abelian subgroup of it, whose existence is rooted in turn in the cohomology of the Galilei group [8]. Remarkably, it is this same cohomology which rules central extensions [9].

In this paper we performed an analogous study in the *Landau problem*, based on the Newton-Hooke group. The clue is that Newton-Hooke and Galilean symmetries are indeed “hiddenly the same” [3], and have therefore identical cohomological structures [12].

The intuitive content of Kohn’s theorem, i.e., the relation between [Newton-Hooke] symmetry and center-of-mass, is now clear : each particle, taken individually, would carry such a symmetry; Kohn’s condition is precisely what is needed to extend this symmetry to the center-of-mass, which will hence represent the motion of all particles collectively.

A method for finding approximate solutions of the 3-body problem of Celestial Mechanics, also used in Galactic Dynamics [27], is referred to as the *Hill Problem*. It was found recently [6] that the latter also has a symmetry reminiscent of the Newton-Hooke one, except for rotations, which are broken.

At the technical level, the Hill Problem is a particular case of an anisotropic harmonic oscillator in an effective magnetic field.

In this paper, we performed a similar study for a *general anisotropic harmonic oscillator*.

All our investigations here have been purely classical. The decomposition of Newton-Hooke symmetry into Heisenberg algebras is, however, particularly useful for the quantum description, see [18, 20] for details.

Appendix A

We find here the canonical form of the 4×4 antisymmetric nonsingular matrix ω undergoing the transformation

$$\omega \rightarrow D\omega D^T, \quad \text{where } D \text{ obeys } D^T X D = X, \quad (\text{A.1})$$

X being the matrix defined in eqn. (5.3).

As it has been noted in the main text, X can be put into canonical form, which depends on the assumption concerning the eigenvalues of X .

Consider first X positive definite; then we can put $X = I$. As a result D is orthogonal and we have to find the canonical form of ω under $\mathfrak{o}(4)$ transformations (A.1). This resembles the problem of classifying the electromagnetic field configurations under the Lorentz group $O(3, 1)$. Guided by this analogy, we define

$$f_i = \omega_{0i}, \quad g_i = \frac{1}{2}\epsilon_{jk}\omega_{jk}. \quad (\text{A.2})$$

Note that f_i and g_i transform like vectors under $SO(3)$ transformations acting on the last three coordinates. Moreover, $\det \omega \sim (\vec{f} \cdot \vec{g})^2$, so that $\vec{f} \cdot \vec{g} \neq 0$, i.e., $\vec{f} \neq 0$, $\vec{g} \neq 0$ and \vec{f} is not perpendicular to \vec{g} .

Let us consider the rotation in the plane spanned by the O -axis, and the axis which is orthogonal to it and defined by the unit vector \vec{n} . The transformation rules under such a rotation read

$$\begin{aligned} \vec{f}'_{\parallel} &= \vec{f}_{\parallel}, & \vec{f}'_{\perp} &= \vec{f}_{\perp} \cos \varphi + (\vec{n} \times \vec{g}_{\perp}) \sin \varphi, \\ \vec{g}'_{\parallel} &= \vec{g}_{\parallel}, & \vec{g}'_{\perp} &= \vec{g}_{\perp} \cos \varphi - (\vec{n} \times \vec{f}_{\perp}) \sin \varphi, \end{aligned} \quad (\text{A.3})$$

where \parallel (\perp) denotes the component parallel (orthogonal) to \vec{n} . If $\vec{f} \nparallel \vec{g}$ we put

$$\vec{n} = \frac{\vec{f} \times \vec{g}}{|\vec{f} \times \vec{g}|} \quad \text{and} \quad \sin 2\varphi = \frac{2|\vec{f} \times \vec{g}|}{\vec{f}^2 + \vec{g}^2} \quad (\text{A.4})$$

to achieve $\vec{f}' \parallel \vec{g}'$. Then by $SO(3)$ rotation one gets further $f_i = \Omega_1 \delta_{i2}$, $\Omega_1 > 0$, $g_i = -\Omega_2 \delta_{i2}$. Renumbering, if necessary, $1 \leftrightarrow 3$ (which is an $O(3)$ transformation) we let $\Omega_2 > 0$. Due to definition (A.2),

$$\omega = \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & g_3 & -g_2 \\ -f_2 & -g_3 & 0 & g_1 \\ -f_3 & g_2 & -g_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Omega_1 & 0 \\ 0 & 0 & 0 & \Omega_2 \\ -\Omega_1 & 0 & 0 & 0 \\ 0 & -\Omega_2 & 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

Consider next the case of semidefinite X with one null eigenvalue. Then X can be put in the form $X = \begin{pmatrix} 0 & 0 \\ 0 & I_3 \end{pmatrix}$. Put

$$D = \begin{pmatrix} d & A \\ B & U \end{pmatrix}. \quad (\text{A.6})$$

Eqns. (A.1) implies $B = 0$, $U \in O(3)$, so D acquires the form $D = \begin{pmatrix} d & A \\ 0 & U \end{pmatrix}$, $d \neq 0$. Then, with $\omega_{ij} = \epsilon_{ijk}g_k$,

$$D\omega D^T = \begin{pmatrix} 0 & dfU^T + A\omega_g U^T \\ -dUf^T + U\omega_g A^T & U\omega_g U^T \end{pmatrix}. \quad (\text{A.7})$$

Here ω_g is an antisymmetric matrix, so it belongs to the algebra $\mathfrak{so}(3)$. One can choose therefore $U \in SO(3)$ such that

$$U\omega_g U^T = \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & 0 \\ -\Omega_2 & 0 & 0 \end{pmatrix}, \quad \Omega_2 > 0. \quad (\text{A.8})$$

Consider now the elements $dfU^T + A\omega_g U^T = dfU^T + AU^T U\omega_g U^T$. Call

$$dfU^T \equiv (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3), \quad AU^T \equiv (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3). \quad (\text{A.9})$$

Then

$$\begin{aligned} dfU^T + AU^T U\omega_g U^T &= (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) + (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & 0 \\ -\Omega_2 & 0 & 0 \end{pmatrix} \\ &= (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) + (-\Omega_2 \tilde{a}_3, 0, \Omega_2 \tilde{a}_1). \end{aligned} \quad (\text{A.10})$$

Knowing f , U and d one determines $\tilde{f}_{1,2,3}$ and chooses $\tilde{a}_{1,3}$ in such a way that

$$dfU^T + AU^T U\omega_g U^T = (0, \tilde{f}_2, 0), \quad \tilde{f}_2 \neq 0. \quad (\text{A.11})$$

By an appropriate choice of d we get $0 < \tilde{f}_2 \equiv \Omega_1$; so (A.7) acquires the form (A.5).

Finally, let X have two positive, one negative and one zero eigenvalue. Without loosing generality, we put

$$X = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}, \quad G \equiv \text{diag}(-1, 1, 1) \quad (\text{A.12})$$

With D of the form (A.6) eqn. (A.1) yields $B = 0$, $U \in O(2, 1)$; $D\omega D^T$ has the same form (A.7).

Consider now $U\omega_g U^T$. Again proceeding along the same lines as in the classification of electromagnetic field configurations, we find that $U\omega_g U^T$ can acquire three “canonical” forms:

$$U\omega_g U^T = \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & 0 \\ -\Omega_2 & 0 & 0 \end{pmatrix}, \quad \Omega_2 > 0 \quad (\text{A.13})$$

$$U\omega_g U^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta \\ 0 & -\Delta & 0 \end{pmatrix}, \quad \Delta > 0 \quad (\text{A.14})$$

$$U\omega_g U^T = \begin{pmatrix} 0 & 0 & \Sigma \\ 0 & 0 & \Sigma \\ -\Sigma & -\Sigma & 0 \end{pmatrix}, \quad \Sigma \neq 0. \quad (\text{A.15})$$

If (A.13) holds the same reasoning as previously leads to eqn. (A.5). In the second case

$$\omega = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta \\ 0 & 0 & -\Delta & 0 \end{pmatrix}, \quad \Omega > 0, \quad \Delta > 0. \quad (\text{A.16})$$

Finally, if A.15 holds,

$$\omega = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ -\Omega & 0 & 0 & \Sigma \\ 0 & 0 & 0 & \Sigma \\ 0 & -\Sigma & -\Sigma & 0 \end{pmatrix}. \quad (\text{A.17})$$

Appendix B

We consider here the orbit method for the Lie algebra (5.1). The general element of the dual space can be written as

$$h\tilde{H} + m\tilde{M} + z_i\tilde{\xi}^i. \quad (\text{B.1})$$

Consider the coadjoint action of $g = \exp(iy^k \xi_k)$. It reads

$$\begin{aligned} m' &= m, \\ z'_i &= z_i + \omega_{ki} y^k m, \\ h' &= h + y^k \omega_{kl} X_{lj} z_j + \frac{1}{2} y^k y^l \omega_{lm} \omega_{kj} X_{mj} m. \end{aligned} \quad (\text{B.2})$$

Assuming $m \neq 0$ and using the fact that ω is invertible, we conclude that each orbit contains the points corresponding to $z_i = 0$. The set of these points forms the coadjoint orbit of the stability subgroup of the relations $z_i = 0$. However, the latter is generated by M and H , so the coadjoint orbits are trivial. We conclude that $z_i = 0$ define exactly one point on coadjoint orbit. Therefore, generating the whole orbit by the action of our group on that point we conclude that the orbit can be parametrized by the variables z_i and

$$h = \epsilon + \frac{1}{2m} X_{ij} z_i z_j, \quad (\text{B.3})$$

where ϵ is the value of h at the point $z_i = 0$ (internal energy). The basic Poisson bracket reads

$$\{z_i, z_j\} = \omega_{ij} m, \quad (\text{B.4})$$

which completes the description.

The additional symmetry generators can be dealt with in a similar way.

In the case of two degrees of freedom and (semi)definite H it is convenient to identify the “physical” generators as described by eqns. (5.8) (i.e. to single out the boosts and momenta). In this basis the counterparts of dual coordinates z_i are denoted by p_i and b_i (cf. eqns. (5.11) and (5.12)).

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- [28] This is particularly clear if we use the equivalent Lagrangian

$$\tilde{L}_{\pm} = \frac{1}{2}\epsilon_{ij}y_i\dot{y}_j \mp \frac{\omega}{2}y_i^2.$$

- [29] For the record, $\mathbf{\Pi}$ and \mathbf{P} are related as $P^i = \mathbf{\Pi}^i + M\omega\varepsilon^{ij}X^j$.
- [30] Note that $dK^i/dt = \partial_t K^i + \{H, K^i\} = 0$ as it should.
- [31] An isotropic harmonic electric force can be freely added, cf. [20])
- [32] In the Hill case $\omega_+ = 0$ and the \mathbf{X}_+ -dynamics is free, while $\omega_- = \omega/\Delta$, cf. [7].
- [33] In fact, $\sigma = \theta/(1 - \theta\tau)$, $\theta = \sigma/(1 + \sigma\tau)$, where θ is the non-commutativity parameter.
- [34] m is the eigenvalue of the operator M .
- [35] The Bargmann space (6.3) is not conformally flat as its Weyl tensor does not vanish, unless $\omega_1 = \omega_2$.